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OUTER PLETHYSM, BURNSIDE RING AND β-RINGS

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Consider the graded ring $B(S)=\bigoplus_{n\geq 0} B(\Sigma_n)$, where $B(\Sigma_n)$ is the Burnside ring of the symmetric group of degree n, Σ_n. Our main aim is to provide $B(S)$ with the structure of operator ring by introducing a third operation, '$\vee$', called outer plethysm, which satisfies the suitable identities. Also, we establish the basic ideas for the study of a β-rings theory.

1. Introduction

The set Λ of all λ-operations (natural transformations from the underlying set functor, from the category of λ-rings to the category of sets, to itself) is a λ-ring [10, Chapter I].

The free λ-ring in one generator, $Z[u_1, u_2, \dots]$ (where $\lambda^k(u_1)=u_k$, $\forall k\geq 1$), and the graded ring $R(S)=\bigoplus_{n\geq 0} R(\Sigma_n)$ (where $R(\Sigma_n)$ is the representation ring of the symmetric group of degree n, Σ_n) are λ-rings isomorphic to Λ [1, 9, 10].

Knutson conjectures in [10] that the operation of $R(S)$ (called outer plethysm) which corresponds with the composition of Λ can be described through a suitable wreath product. Hoffman proves and describes this connection in [9].

Now, consider the abelian group $B(S)=\bigoplus_{n\geq 0} B(\Sigma_n)$, where $B(\Sigma_n)$ is the Burnside ring of the group Σ_n (see [6]); then, $B(S)$ becomes a graded ring with the cross product, '$\underline{\times}$', defined in Section 4.

In this paper we provide $B(S)$ with the structure of *operator ring* by defining a third operation, '$\vee$' (outer plethysm), which satisfies the identities of Definition 4.4. Observe that '$\vee$' will not be right additive, therefore we have to define '$b\vee a$' for every a in $B(S)$ and not merely for an 'a' in some $B(\Sigma_n)$. We use only algebraic techniques, avoiding the topological description of the Burnside ring given by Dieck [4].

Similar results to Lemmas 4.7 and 4.9 can be found in Brennan's thesis [13, Chapter III], and they have been included here in order to make this paper more selfcontained.

Finally, in Section 5 we point out some notes about β-rings.

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Notation

In general, we follow the notation of [9], but we write $R(S)$ instead of S.

If $\psi:\Omega\to\Gamma$ is a group homomorphism, ψ^* (resp. ψ_*) will be the restriction (resp. induction) from $R(\Gamma)$ or $B(\Gamma)$ to $R(\Omega)$ or $B(\Omega)$ (resp. from $R(\Omega)$ or $B(\Omega)$ to $R(\Gamma)$ or $B(\Gamma)$).

The existence and properties of restriction and induction maps in the Burnside ring of a finite group are recalled in Section 3.

The wreath product $\Gamma\sim\Sigma_k$ (natural action) will be denoted by $\Sigma_k\langle\Gamma\rangle$. Also, we use the symbol '$\underline{\times}$' to denote the cross product instead of '$\times$'.

$\mathbb{N}$ is the set of nonnegative integers. For each k in $\mathbb{N}$, an r-partition of k is a sequence $(i_1,\dots,i_r)$ such that i_j is in $\mathbb{N}$ for $j=1,\dots,r$ and $i_1+\cdots+i_r=k$.

In Section 2 we identify the rings $R(\Gamma_1\times\Gamma_2)$ and $R(\Gamma_1)\otimes R(\Gamma_2)$, for all finite groups Γ_1 and Γ_2.

2. The operator ring $R(S)$

As we have already recalled, the operation 'composition' of Λ is transferred to $R(S)$ through the isomorphism $\Lambda\simeq R(S)$, and then $R(S)$ becomes an operator ring. Hoffman [9, p. 28–29] proves that the behaviour of this operation '$\vee$' for elements $b\in R(\Sigma_k)$, $a\in R(\Sigma_l)$ is

$$b\vee a=\theta_*[(\bar{\tau}_k(a))(\pi^*(b))]$$

where the group homomorphisms

$$\theta:\Sigma_k\langle\Sigma_l\rangle\to\Sigma_{kl},\qquad \pi:\Sigma_k\langle\Sigma_l\rangle\to\Sigma_k$$

are the inclusion and the projection, respectively, and the map

$$\bar{\tau}_k:R(\Sigma_l)\to R(\Sigma_k\langle\Sigma_l\rangle)$$

acts on actual representations by

$$\bar{\tau}_k(M)=M\otimes\overset{k}{\cdots}\otimes M$$

with the obvious action of $\Sigma_k\langle\Sigma_l\rangle$ (see [9, p. 12, ...]).

Up to now, '$b\vee a$' has not been described when $a=a_1+\cdots+a_r$ with a_j in $R(\Sigma_{l_j})$ for $j=1,\dots,r$ (there is no problem concerning b due to the left additivity of '$\vee$'). In order to suggest the 'good' definition of the operation '$\vee$' in $B(S)$ we give such a description, of which the proof is omitted.

Proposition 2.1. *Let $b\in R(\Sigma_k)$, $a_j\in R(\Sigma_{l_j})$ $(j=1,\dots,r)$. Then*

$$b\vee(a_1+\cdots+a_r)=\sum_{\alpha}(\theta_\alpha)_*\left[\left(\bigotimes_j\bar{\tau}_{i_j}(a_j)\right)(\pi_\alpha^*\delta_\alpha^*(b))\right]\tag{1}$$

where the summation is over r-partitions $\alpha=(i_1,\dots,i_r)$ of k; θ_α is the composition

$$\Sigma_{i_1}\langle\Sigma_{l_1}\rangle\times\cdots\times\Sigma_{i_r}\langle\Sigma_{l_r}\rangle\to\Sigma_{i_1l_1}\times\cdots\times\Sigma_{i_rl_r}\to\Sigma_{i_1l_1+\cdots+i_rl_r},$$

$$\bigotimes_j \bar{\tau}_{i_j}(a_j):=\bar{\tau}_{i_1}(a_1)\otimes\cdots\otimes\bar{\tau}_{i_r}(a_r)\in R(\Sigma_{i_1}\langle\Sigma_{l_1}\rangle\times\cdots\times\Sigma_{i_r}\langle\Sigma_{l_r}\rangle),$$

$$\pi_\alpha:\Sigma_{i_1}\langle\Sigma_{l_1}\rangle\times\cdots\times\Sigma_{i_r}\langle\Sigma_{l_r}\rangle\to\Sigma_{i_1}\times\cdots\times\Sigma_{i_r},$$

$$\delta_\alpha:\Sigma_{i_1}\times\cdots\times\Sigma_{i_r}\to\Sigma_k.\qquad\square$$

3. Restriction and induction in the Burnside ring

If Γ is a finite group, by $B(\Gamma)$ we shall denote the Burnside ring of Γ as defined by Dress [6]; this is the Grothendieck ring of the semiring of the isomorphism classes of finite Γ-sets, where the sum is defined by the disjoint union and the product by the Cartesian product with diagonal action. If X is a Γ-set, we shall denote also by X the corresponding element in $B(\Gamma)$.

Induction 3.1. Let $\phi:\Gamma\to\Omega$ be a group homomorphism, X a Γ-set. In the Cartesian product $\Omega\times X$ we define the equivalence relation $\sim$ given by $(g,x)\sim(g_1,x_1)$ if and only if there exists h in Γ such that $g_1=g\phi(h^{-1})$ and $x_1=hx$.

We shall denote the equivalence class of (g,x) by $(\overline{g,x})$.

The set $\Omega\times_\phi X=\Omega\times X/\sim$ is an Ω-set with the action

$$g_1(\overline{g,x})=(\overline{g_1g,x}),\quad \forall g_1,g\in\Omega, x\in X$$

and it is called the Ω-set induced from X by ϕ, and denoted by $\phi_*(X)$.

By additivity, ϕ_* is extended to a homomorphism of abelian groups

$$\phi_*:B(\Gamma)\to B(\Omega).$$

Restriction 3.2. Let $\phi:\Gamma\to\Omega$ be a group homomorphism and X an Ω-set. Then X becomes a Γ-set, denoted by $\phi^*(X)=X_\phi$, with respect to the action

$$hx:=\phi(h)x,\quad \forall h\in\Gamma.$$

ϕ^* is extended to a homomorphism of abelian groups

$$\phi^*:B(\Omega)\to B(\Gamma).$$

Proposition 3.3. *Let $\phi:\Gamma\to\Omega$ and $\psi:\Omega\to\Gamma$ be homomorphisms of finite groups. Then*

(i) *ϕ^* is a homomorphism of commutative rings with unit*;

(ii) $(\psi\phi)^*=\phi^*\psi^*$;

(iii) $(\psi\phi)_*=\psi_*\phi_*$;

(iv) *If* ϕ *is an isomorphism, then* $\phi_*\phi^*=\mathrm{id}_{B(\Omega)}$, $\phi^*\phi_*=\mathrm{id}_{B(\Gamma)}$;
(v) *If* $\Gamma=\Omega$ *and* ϕ *is an inner automorphism, then* $\phi^*=\phi_*=\mathrm{id}_{B(\Gamma)}$. □

Theorem 3.4 (Mackey). *If* $\alpha:\Gamma\to\Lambda$ *and* $\beta:\Omega\to\Lambda$ *are inclusions of subgroups, then*

$$\beta^*\alpha_*=\sum_g(\beta_g)^*(\alpha_g)_*$$

where g ranges over a set of (Γ,Ω)*-double coset representatives,* $\alpha_g:\Gamma^g\cap\Omega\to\Gamma$ *is given by* $\alpha_g(x)=gxg^{-1}$, *and* $\beta_g:\Gamma^g\cap\Omega\to\Omega$ *is the inclusion.* □

Theorem 3.5 (Frobenius). *Let* $\alpha:\Gamma\to\Omega$ *be a group homomorphism. For every x in* $B(\Gamma)$ *and every y in* $B(\Omega)$, *one has*

$$\alpha_*(x)\cdot y=\alpha_*(x\cdot\alpha^*(y)).\qquad\square$$

The proofs of Proposition 3.3 and Theorems 3.4 and 3.5 are standard. Dress comments the existence of these properties in [7, p. 42] in connection with the idea of $B(-)$ as G-functor [8].

Another property that we will need is the following:

Proposition 3.6. *If the diagram is a pullback of finite groups, then* $\pi_1^*\alpha_*=\beta_*\pi^*$.

$$\begin{tikzcd} \Omega \arrow[r, "\beta"] \arrow[d, "\pi"'] & \Omega_1 \arrow[d, "\pi_1"] \\ \Gamma \arrow[r, "\alpha"] & \Gamma_1 \end{tikzcd}\qquad\square$$

The proof is a routine checking.

4. Operator ring structure in $B(S)$

As we have already mentioned, $B(S)=\bigoplus_{k\geq 0}B(\Sigma_k)$ is an abelian group. The following ring homomorphism will be very useful:

Definition 4.1. If Γ_1,Γ_2 are finite groups, and (for $i=1,2$) $\pi_i:\Gamma_1\times\Gamma_2\to\Gamma_i$ denote the projections, the homomorphism $\mu:B(\Gamma_1)\otimes B(\Gamma_2)\to B(\Gamma_1\times\Gamma_2)$ is defined as the composition of $\pi_1^*\otimes\pi_2^*$ followed by the juxtaproduct

$$B(\Gamma_1\times\Gamma_2)\otimes B(\Gamma_1\times\Gamma_2)\to B(\Gamma_1\times\Gamma_2).$$

This can be done in the obvious way for more than two factors. Sometimes the notation $x\otimes y$ will be used instead of $\mu(x\otimes y)$.

Let Γ be a finite group and $P=\prod_{k\geq 0} B(\Sigma_k\langle\Gamma\rangle)$. The definition of the cross product, $\underline{\times}$, is (analogous to [9, p. 6]):

$$P\otimes P \xrightarrow{\underline{\times}} P$$

with components

$$B(\Sigma_i\langle\Gamma\rangle)\otimes B(\Sigma_j\langle\Gamma\rangle) \xrightarrow{\mu} B(\Sigma_i\langle\Gamma\rangle\times\Sigma_j\langle\Gamma\rangle) \xrightarrow{\phi_*} B(\Sigma_{i+j}\langle\Gamma\rangle)$$

where $\phi:\Sigma_i\langle\Gamma\rangle\times\Sigma_j\langle\Gamma\rangle\to\Sigma_{i+j}\langle\Gamma\rangle$ is the obvious group homomorphism.

With this multiplication $B(S)$ becomes a graded ring ($\Gamma=1$).

Also, we define the multiplication $\circledast$ on the group $\bigoplus_{l=0}^{\infty} B(\Sigma_k\langle\Sigma_l\langle\Gamma\rangle\rangle)$ (see [9, p. 8]) given on components by the commutativity of the diagram

$$\begin{tikzcd}
B(\Sigma_k\langle\Sigma_i\langle\Gamma\rangle\rangle)\otimes B(\Sigma_k\langle\Sigma_j\langle\Gamma\rangle\rangle) \arrow[r, "\circledast"] \arrow[d, "\mu"'] & B(\Sigma_k\langle\Sigma_{i+j}\langle\Gamma\rangle\rangle) \\
B(\Sigma_k\langle\Sigma_i\langle\Gamma\rangle\rangle\times\Sigma_k\langle\Sigma_j\langle\Gamma\rangle\rangle) \arrow[r, "\delta^*"] & B(\Sigma_k\langle(\Sigma_i\times\Sigma_j)\langle\Gamma\rangle\rangle) \arrow[u, "\alpha_*"']
\end{tikzcd}$$

where μ is given in Definition 4.1 and

$$\delta:\Sigma_k\langle(\Sigma_i\times\Sigma_j)\langle\Gamma\rangle\rangle\to\Sigma_k\langle\Sigma_i\langle\Gamma\rangle\rangle\times\Sigma_k\langle\Sigma_j\langle\Gamma\rangle\rangle,$$
$$\alpha:\Sigma_k\langle(\Sigma_i\times\Sigma_j)\langle\Gamma\rangle\rangle\to\Sigma_k\langle\Sigma_{i+j}\langle\Gamma\rangle\rangle$$

are the obvious group homomorphisms.

Now, we define the map T from the set of isomorphism classes of simple Γ-sets to P, with components T_k given by

$$T_k(X)=X\times\overset{k}{\cdots}\times X=X^k.$$

$T_k(X)$ is a $\Sigma_k\langle\Gamma\rangle$-set with the action

$$(g_1,\dots,g_k;\sigma)(x_1,\dots,x_k)=(g_1x_{\sigma^{-1}(1)},\dots,g_kx_{\sigma^{-1}(k)}).$$

Proposition 4.2 (see [9, (3.7) and p. 112]). *T can be extended to a homomorphism from the additive group $B(S)$ to the group P with the cross product.* (*See* [9, (1.6) *and* (2.3)]).

Moreover, T satisfies the following properties:

(o) *For every Γ-set Y, $T_k(Y)=Y\times\cdots\times Y$ with the same action as above*;

(i) *$T_0(x)=1$ and $T_1(x)=x$, for every $x\in B(\Gamma)$*;

(ii) *$T(x+y)=T(x)\underline{\times}T(y)$, for every $x,y\in B(\Gamma)$*;

(iii) *The following diagram commutes*:

$$\begin{tikzcd} B(\Gamma) \arrow[r, "T^{\otimes r}"'] \arrow[d, "T"'] & P\otimes\cdots\otimes P \arrow[d, "\bar{\mu}"] \\ P \arrow[r, "\Delta^{(r)}"] & \prod B(\Sigma_{k_1}\langle\Gamma\rangle\times\cdots\times\Sigma_{k_r}\langle\Gamma\rangle) \end{tikzcd}$$

where the product ranges over all the $k_1,\ldots,k_r\ge 0$; $\bar{\mu}$ *has components*

$$B(\Sigma_{k_1}\langle\Gamma\rangle)\otimes\cdots\otimes B(\Sigma_{k_r}\langle\Gamma\rangle)\xrightarrow{\mu} B(\Sigma_{k_1}\langle\Gamma\rangle\times\cdots\times\Sigma_{k_r}\langle\Gamma\rangle)$$

and $\Delta^{(r)}, \Delta^{(r)}_{k_1,\ldots,k_r}=\alpha^*_{k_1,\ldots,k_r}$ *defined from*

$$\alpha_{k_1,\ldots,k_r}:\Sigma_{k_1}\langle\Gamma\rangle\times\cdots\times\Sigma_{k_r}\langle\Gamma\rangle\to\Sigma_k\langle\Gamma\rangle \quad (k=k_1+\cdots+k_r);$$

(iv) $T(xy)=T(x)T(y)$ *for every* $x,y\in B(\Gamma)$;

(v) *The following diagram commutes*:

$$\begin{tikzcd} B(\Gamma) \arrow[r, "T"'] \arrow[d, "T"'] & P \arrow[d, "\zeta"] \\ P \arrow[r, "\square"] & \prod\limits_{k,l\ge 0} B(\Sigma_k\langle\Sigma_l\langle\Gamma\rangle\rangle) \end{tikzcd}$$

where ζ *has components*

$$\zeta_{k,l}:B(\Sigma_l\langle\Gamma\rangle)\to B(\Sigma_k\langle\Sigma_l\langle\Gamma\rangle\rangle)$$

which act on $B(\Sigma_l\langle\Gamma\rangle)$ *in the same way as* T_k ($\Sigma_l\langle\Gamma\rangle$ *playing the role of* Γ). *We will sometimes denote* $\zeta_{k,l}=T_k$.

$\square$ *has components* $\square_{k,l}=\theta^*_{k,l}$, *where*

$$\theta_{k,l}:\Sigma_k\langle\Sigma_l\langle\Gamma\rangle\rangle\to\Sigma_{kl}\langle\Gamma\rangle$$

is defined by

$$\begin{aligned}&\theta_{k,l}[(g_{11},\ldots,g_{1l};\sigma_1),\ldots,(g_{k1},\ldots,g_{kl};\sigma_k);\tau]\\ &\quad=[g_{11},\ldots,g_{1l},\ldots,g_{k1},\ldots,g_{kl};(\sigma_1,\ldots,\sigma_k;\tau)].\end{aligned}$$

Sketch of proof. (o), (i) and (ii) are not difficult.

(iii) It is enough to prove that both $\mu T^{\otimes r}$ and $\Delta^{(r)}T$ coincide on Γ-sets, their images are contained in the set of $\underline{\times}$-invertibles of $B(\Sigma_{k_1}\langle\Gamma\rangle\times\cdots\times\Sigma_{k_r}\langle\Gamma\rangle)$, and are homomorphisms from '+' to '$\underline{\times}$'; then proceed in the same way as the proof of [9, (3.7) III].

(iv) Check the statement for X, Y Γ-sets and then follow the proof of [9, Proposition (3.7) IV].

(v) The diagram commutes for Γ-sets. Now, the proof is similar to [9, (3.7) V]. $\square$

Lemma 4.3. *Let* $a \in B(\Sigma_i\langle\Gamma\rangle)$, $b \in B(\Sigma_j\langle\Gamma\rangle)$. *Then*

$$T_k(a \underline{\times} b) = T_k(a) \circledast T_k(b).$$

Proof. Follow step by step the proof of [9, Proposition (2.10)]. □

Our main aim is to provide $B(S)$ with the structure of operator ring. Let us recall what this means:

Definition 4.4. An *operator ring* is a commutative ring R with identity 1, and with a chosen element e ($\neq 1$) of R and a composition $\vee : R \times R \to R$ satisfying:

(1) $(x_1 + x_2) \vee x_3 = (x_1 \vee x_3) + (x_2 \vee x_3)$,

(2) $(x_1 x_2) \vee x_3 = (x_1 \vee x_3)(x_2 \vee x_3)$,

(3) $(x_1 \vee x_2) \vee x_3 = x_1 \vee (x_2 \vee x_3)$,

(4) $x_1 \vee e = x_1$,

for all x_1, x_2, x_3 in R.

We want to define the operation '$\vee$' in $B(S)$. Let $k, l_1, \ldots, l_r$ be elements in $\mathbb{N}$. For every r-partition of k, $\alpha = (i_1, \ldots, i_r)$, one has the group homomorphisms θ_α, δ_α and π_α mentioned in Proposition 2.1.

Definition 4.5. Let $b \in B(\Sigma_k)$, $a_j \in B(\Sigma_{l_j})$ $(j = 1, \ldots, r)$. We define the composition '$\vee$' in $B(S)$ by

$$b \vee (a_1 + \cdots + a_r) = \sum_\alpha (\theta_\alpha)_* \left[\left(\bigotimes_{j=1}^{r} T_{i_j}(a_j) \right) (\pi_\alpha^* \delta_\alpha^*(b)) \right]$$

where the sum ranges over all r-partitions of k: $\alpha = (i_1, \ldots, i_r)$.

In some cases an element a in $B(S)$ will not be represented as a sum $a = a_1 + \cdots + a_r$ with each summand a_j appearing in a different $B(\Sigma_{l_j})$; so we have to prove that this composition does not depend on the representation of a.

Proposition 4.6. *The operation composition '$\vee$' of Definition* 4.5 *is well defined.*

Proof. Let $a_1 = c_1 + d_1 \in B(\Sigma_{l_1})$, $a_j \in B(\Sigma_{l_j})$ $(j = 2, \ldots, r)$. Put $l_0 := l_1$, $a_0' := c_1$, $a_1' := d_1$, $a_j' := a_j$ $(j = 2, \ldots, r)$. For every $(r+1)$-partition of k, $\alpha' = (i_0', \ldots, i_r')$, consider the r-partition of k, $\alpha = (i_0' + i_1', i_2', \ldots, i_r') = (i_1, i_2, \ldots, i_r)$.

Let $\phi_{\alpha'}$ be the group homomorphism

$$\Sigma_{i_0'}\langle\Sigma_{l_0}\rangle \times \Sigma_{i_1'}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_r'}\langle\Sigma_{l_r}\rangle \to \Sigma_{i_0' + i_1'}\langle\Sigma_{l_1}\rangle \times \Sigma_{i_2'}\langle\Sigma_{l_2}\rangle \times \cdots \times \Sigma_{i_r'}\langle\Sigma_{l_r}\rangle$$

and $\theta_{\alpha'}, \theta_\alpha, \delta_{\alpha'}, \delta_\alpha, \pi_{\alpha'}, \pi_\alpha$ as before. It is easy to check that

$$\theta_{\alpha'} = \theta_\alpha \phi_{\alpha'} \quad \text{and} \quad \delta_{\alpha'} \pi_{\alpha'} = \delta_\alpha \pi_\alpha \phi_{\alpha'}.$$

So, we have $(\theta_{\alpha'})_* = (\theta_\alpha)_*(\phi_{\alpha'})_*$ and $\pi_{\alpha'}^*\delta_{\alpha'}^* = \phi_{\alpha'}^*\pi_\alpha^*\delta_\alpha^*$.

Now,

$$\begin{aligned}
&b \vee (a_0' + a_1' + \cdots + a_r')\\
&= \sum_{\alpha'} (\theta_{\alpha'})_* \left[\left(\bigotimes_{j=0}^{r} T_{i_j'}(a_j')\right)(\pi_{\alpha'}^*\delta_{\alpha'}^*(b))\right]\\
&= \sum_{\alpha'} (\theta_\alpha)_*(\phi_{\alpha'})_* \left[\left(\bigotimes_{j=0}^{r} T_{i_j'}(a_j')\right)(\phi_{\alpha'}^*\pi_\alpha^*\delta_\alpha^*(b))\right]
\end{aligned}$$

by Frobenius reciprocity

$$= \sum_{\alpha'} (\theta_\alpha)_* \left[\left((\phi_{\alpha'})_*\left(\bigotimes_{j=0}^{r} T_{i_j'}(a_j')\right)\right)(\pi_\alpha^*\delta_\alpha^*(b))\right]$$

when α' ranges over all $(r+1)$-partitions of k, α ranges over all r-partitions of k. Put $P(\alpha)$ for the set of $(r+1)$-partitions α' defining the same α

$$\begin{aligned}
&= \sum_{\alpha} (\theta_\alpha)_* \left[\left(\left[\sum_{P(\alpha)} (T_{i_0'}(a_0') \underline{\times} T_{i_1'}(a_1'))\right] \otimes T_{i_2'}(a_2') \otimes \cdots \otimes T_{i_r'}(a_r')\right)(\pi_\alpha^*\delta_\alpha^*(b))\right]\\
&= \sum_{\alpha} (\theta_\alpha)_*[(T_{i_1}(a_0' + a_1') \otimes T_{i_2}(a_2') \otimes \cdots \otimes T_{i_r}(a_r'))(\pi_\alpha^*\delta_\alpha^*(b))]\\
&= b \vee (a_1 + \cdots + a_r). \qquad \square
\end{aligned}$$

Lemma 4.7 (see [9, Theorem (1.2)] and [3, Chapter III]). *Let* $n = \varepsilon_1 + \cdots + \varepsilon_r = \omega_1 + \cdots + \omega_s$ *with* $\varepsilon_i \in \mathbb{N}$ $(i = 1, \ldots, r)$ *and* $\omega_j \in \mathbb{N}$ $(j = 1, \ldots, s)$. *Let* H, K *be the subgroups of* Σ_n:

$$H = \Sigma_{\varepsilon_1} \times \cdots \times \Sigma_{\varepsilon_r}, \qquad K = \Sigma_{\omega_1} \times \cdots \times \Sigma_{\omega_s}.$$

Let M *be the set of all* $\mathbb{N}$*-matrices* $r \times s$ *with row sums* $(\varepsilon_1, \ldots, \varepsilon_r)$ *and column sums* $(\omega_1, \ldots, \omega_s)$. *Let* $C = \{HgK : g \in \Sigma_n\}$.

Then there exists a bijective map $\Psi : M \to C$ *such that if* $\Psi[A = (a_{ij})] = HgK$, *then*

$$H^g \cap K = (\Sigma_{a_{11}} \times \cdots \times \Sigma_{a_{r1}}) \times \cdots \times (\Sigma_{a_{1s}} \times \cdots \times \Sigma_{a_{rs}}) \leq \Sigma_n.$$

Proof. Omitted. $\square$

Notations for Proposition 4.8

$k = k_1 + k_2$.
$\varepsilon = (i_1, \ldots, i_r)$ is an r-partition of k.
$\varepsilon_1 = (i_{11}, \ldots, i_{1r})$ is an r-partition of k_1.
$\varepsilon_2 = (i_{21}, \ldots, i_{2r})$ is an r-partition of k_2.
$\phi \;: \Sigma_{k_1} \times \Sigma_{k_2} \to \Sigma_k$.
$\delta_{\varepsilon_1} : \Sigma_{i_{11}} \times \cdots \times \Sigma_{i_{1r}} \to \Sigma_{k_1}$.
$\delta_{\varepsilon_2} : \Sigma_{i_{21}} \times \cdots \times \Sigma_{i_{2r}} \to \Sigma_{k_2}$.
$\delta_\varepsilon \;: \Sigma_{i_1} \times \cdots \times \Sigma_{i_r} \to \Sigma_k$.

$\pi_{\varepsilon_1} : \Sigma_{i_{11}}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_{1r}}\langle\Sigma_{l_r}\rangle \to \Sigma_{i_{11}} \times \cdots \times \Sigma_{i_{1r}}$.
$\pi_{\varepsilon_2} : \Sigma_{i_{21}}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_{2r}}\langle\Sigma_{l_r}\rangle \to \Sigma_{i_{21}} \times \cdots \times \Sigma_{i_{2r}}$.
$\pi_{\varepsilon} \ : \Sigma_{i_1}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_r}\langle\Sigma_{l_r}\rangle \to \Sigma_{i_1} \times \cdots \times \Sigma_{i_r}$.
$\theta_{\varepsilon_1} : \Sigma_{i_{11}}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_{1r}}\langle\Sigma_{l_r}\rangle \to \Sigma_{i_{11}l_1} \times \cdots \times \Sigma_{i_{1r}l_r} \to \Sigma_{i_{11}l_1 + \cdots + i_{1r}l_r}$.
$\theta_{\varepsilon_2} : \Sigma_{i_{21}}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_{2r}}\langle\Sigma_{l_r}\rangle \to \Sigma_{i_{21}l_1} \times \cdots \times \Sigma_{i_{2r}l_r} \to \Sigma_{i_{21}l_1 + \cdots + i_{2r}l_r}$.
$\alpha \ : (\Sigma_{i_{11}} \times \Sigma_{i_{21}}) \times \cdots \times (\Sigma_{i_{1r}} \times \Sigma_{i_{2r}}) \to \Sigma_{i_1} \times \cdots \times \Sigma_{i_r}$.
$\bar{\alpha} \ : (\Sigma_{i_{11}} \times \Sigma_{i_{21}})\langle\Sigma_{l_1}\rangle \times \cdots \times (\Sigma_{i_{1r}} \times \Sigma_{i_{2r}})\langle\Sigma_{l_r}\rangle \to \Sigma_{i_1}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_r}\langle\Sigma_{l_r}\rangle$.
$\bar{\alpha}_j \ : (\Sigma_{i_{1j}} \times \Sigma_{i_{2j}})\langle\Sigma_{l_j}\rangle \to \Sigma_{i_j}\langle\Sigma_{l_j}\rangle \ (j=1,\ldots,r)$.
$\bar{\pi} \ : (\Sigma_{i_{11}} \times \Sigma_{i_{21}})\langle\Sigma_{l_1}\rangle \times \cdots \times (\Sigma_{i_{1r}} \times \Sigma_{i_{2r}})\langle\Sigma_{l_r}\rangle \to (\Sigma_{i_{11}} \times \Sigma_{i_{21}}) \times \cdots \times (\Sigma_{i_{1r}} \times \Sigma_{i_{2r}})$.
$\gamma \ : (\Sigma_{i_{11}} \times \Sigma_{i_{21}}) \times \cdots \times (\Sigma_{i_{1r}} \times \Sigma_{i_{2r}}) \to (\Sigma_{i_{11}} \times \cdots \times \Sigma_{i_{1r}}) \times (\Sigma_{i_{21}} \times \cdots \times \Sigma_{i_{2r}})$.
$\bar{\gamma} \ : (\Sigma_{i_{11}} \times \Sigma_{i_{21}})\langle\Sigma_{l_1}\rangle \times \cdots \times (\Sigma_{i_{1r}} \times \Sigma_{i_{2r}})\langle\Sigma_{l_r}\rangle$
$\quad \to (\Sigma_{i_{11}}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_{1r}}\langle\Sigma_{l_r}\rangle) \times (\Sigma_{i_{21}}\langle\Sigma_{l_1}\rangle \times \cdots \times \Sigma_{i_{2r}}\langle\Sigma_{l_r}\rangle)$.
$\psi \ : \Sigma_{i_{11}l_1 + \cdots + i_{1r}l_r} \times \Sigma_{i_{21}l_1 + \cdots + i_{2r}l_r} \to \Sigma_{i_1 l_1 + \cdots + i_r l_r}$.
$\sigma \ : \Sigma_{i_1 l_1 + \cdots + i_r l_r} \to \Sigma_{i_1 l_1 + \cdots + i_r l_r}$ is a certain inner automorphism.

Proposition 4.8. *Property* 2 *of Definition* 4.4 *is valid in* $B(S)$.

Proof. We have to prove $(b_1 \underline{\times} b_2) \vee a = (b_1 \vee a) \underline{\times} (b_2 \vee a)$ for all $b_1 b_2, a$ in $B(S)$. By bi-additivity of $\underline{\times}$ and left additivity of '$\vee$' we may assume $b_1 \in B(\Sigma_{k_1}), b_2 \in B(\Sigma_{k_2})$.
Let $a = a_1 + \cdots + a_r$ with $a_j \in B(\Sigma_{l_j})$ $(j=1,\ldots,r)$.

$$(b_1 \underline{\times} b_2) \vee a = \phi_*(b_1 \otimes b_2) \vee a$$

$$= \sum_{\varepsilon} (\theta_\varepsilon)_* \left[\left(\bigotimes_j T_{i_j}(a_j) \right) (\pi_\varepsilon^* \delta_\varepsilon^* (\phi_*(b_1 \otimes b_2))) \right] =$$

by the Mackey theorem and Lemma 4.7

$$= \sum_{\varepsilon} (\theta_\varepsilon)_* \left[\left(\bigotimes_j T_{i_j}(a_j) \right) \left(\pi_\varepsilon^* \left[\sum_{\varepsilon_1 + \varepsilon_2 = \varepsilon} \alpha_* \gamma^* (\delta_{\varepsilon_1} \otimes \delta_{\varepsilon_2})^* (b_1 \otimes b_2) \right] \right) \right]$$

by applying Proposition 3.6 to $\pi_\varepsilon \bar{\alpha} = \alpha \bar{\pi}$ and putting $\varepsilon := \varepsilon_1 + \varepsilon_2$

$$= \sum_{\varepsilon_1, \varepsilon_2} (\theta_\varepsilon)_* \left[\left(\bigotimes_j T_{i_j}(a_j) \right) (\bar{\alpha}_* \bar{\pi}^* \gamma^* (\delta_{\varepsilon_1}^*(b_1) \otimes \delta_{\varepsilon_2}^*(b_2))) \right]$$

since $\gamma \bar{\pi} = (\pi_{\varepsilon_1} \otimes \pi_{\varepsilon_2}) \bar{\gamma}$

$$= \sum (\theta_\varepsilon)_* \left[\left(\bigotimes_j T_{i_j}(a_j) \right) (\bar{\alpha}_* \bar{\gamma}^* [\pi_{\varepsilon_1}^* \delta_{\varepsilon_1}^*(b_1) \otimes \pi_{\varepsilon_2}^* \delta_{\varepsilon_2}^*(b_2)]) \right] =$$

Now, $\bar{\alpha}^*(\bigotimes_j T_{i_j}(a_j)) = \bigotimes_j \bar{\alpha}_j(T_{i_j}(a_j)) = \bigotimes_j (T_{i_{1j}}(a_j) \otimes T_{i_{2j}}(a_j))$ (the last equality is due to Proposition 4.2(iii)). By Frobenius reciprocity

$$= \sum (\theta_\varepsilon)_* \bar{\alpha}_* \left[\left(\bigotimes_j [T_{i_{1j}}(a_j) \otimes T_{i_{2j}}(a_j)] \right) (\bar{\gamma}^* [\pi_{\varepsilon_1}^* \delta_{\varepsilon_1}^*(b_1) \otimes \pi_{\varepsilon_2}^* \delta_{\varepsilon_2}^*(b_2)]) \right]$$

since $\theta_\varepsilon \bar{\alpha} = \sigma \psi (\theta_{\varepsilon_1} \otimes \theta_{\varepsilon_2}) \bar{\gamma}$ and by Frobenius reciprocity

$$= \sum \sigma_* \psi_* (\theta_{\varepsilon_1} \otimes \theta_{\varepsilon_2})_* \left[\bar{\gamma}_* \left(\bigotimes_j [T_{i_{1j}}(a_j) \otimes T_{i_{2j}}(a_j)] \right) (\pi^*_{\varepsilon_1} \delta^*_{\varepsilon_1}(b_1) \otimes \pi^*_{\varepsilon_2} \delta^*_{\varepsilon_2}(b_2)) \right]$$

since $\sigma_* = 1$ by Proposition 3.3(v)

$$= \sum \psi_* (\theta_{\varepsilon_1} \otimes \theta_{\varepsilon_2})_* \left[\left(\left[\bigotimes_j T_{i_{1j}}(a_j) \right] \right.\right.$$

$$\left.\left. \otimes \left[\bigotimes_j T_{i_{2j}}(a_j) \right] \right) (\pi^*_{\varepsilon_1} \delta^*_{\varepsilon_1}(b_1) \otimes \pi^*_{\varepsilon_2} \delta^*_{\varepsilon_2}(b_2)) \right]$$

$$= \sum \psi_* \left[(\theta_{\varepsilon_1})_* \left(\left[\bigotimes_j T_{i_{1j}}(a_j) \right] [\pi^*_{\varepsilon_1} \delta^*_{\varepsilon_1}(b_1)] \right) \right.$$

$$\left. \otimes (\theta_{\varepsilon_2})_* \left(\left[\bigotimes_j T_{i_{2j}}(a_j) \right] [\pi^*_{\varepsilon_2} \delta^*_{\varepsilon_2}(b_2)] \right) \right]$$

$$= \left(\sum_{\varepsilon_1} (\theta_{\varepsilon_1})_* \left[\left(\bigotimes_j T_{i_{1j}}(a_j) \right) (\pi^*_{\varepsilon_1} \delta^*_{\varepsilon_1}(b_1)) \right] \right)$$

$$\underline{\times} \left(\sum_{\varepsilon_2} (\theta_{\varepsilon_2})_* \left[\left(\bigotimes_j T_{i_{2j}}(a_j) \right) (\pi^*_{\varepsilon_2} \delta^*_{\varepsilon_2}(b_2)) \right] \right)$$

$$= (b_1 \vee a) \underline{\times} (b_2 \vee a). \qquad \square$$

Lemma 4.9. *Let $H = \Sigma_{\varepsilon_1} \langle \Sigma_{l_1} \rangle$ and $K = \Sigma_{a_{11}} \times \cdots \times \Sigma_{a_{1s}}$, with $a_{11} + \cdots + a_{1s} = \varepsilon_1 l_1$, be subgroups of $\Sigma_{\varepsilon_1 l_1}$.*

In the set of all $\mathbb{N}$-matrices $\varepsilon_1 \times s$ with row sums $(l_1, \ldots, l_1)$ and column sums $(a_{11}, \ldots, a_{1s})$ define the equivalence relation $\sim$ by $B_{\varepsilon_1} \sim B'_{\varepsilon_1}$ if and only if there exists a certain permutation of the ε_1 rows of B_{ε_1} to obtain B'_{ε_1}. The equivalence class of B_{ε_1} is written $[B_{\varepsilon_1}]$.

Let M be the set of all equivalence classes and $C = \{HgK : g \in \Sigma_{\varepsilon_1 l_1}\}$. Then there exists a bijective map

$$\Psi : M \to C$$

such that, if $\Psi([B_{\varepsilon_1}]) = HgK$ with $B_{\varepsilon_1} = (b_{ij})$, then $H \cap K^{g^{-1}} = \Sigma_{e_1} \langle \Sigma_{\gamma(1)^1_1} \times \cdots \times \Sigma_{\gamma(1)^1_s} \rangle \times \cdots \times \Sigma_{e_{p_1}} \langle \Sigma_{\gamma(1)^{p_1}_1} \times \cdots \times \Sigma_{\gamma(1)^{p_1}_s} \rangle$ where $\{\gamma(1)^1, \ldots, \gamma(1)^{p_1}\}$ is the set of all s-partitions of l_1: $\gamma(1)^i = (\gamma(1)^i_1, \ldots, \gamma(1)^i_s)$ $(i = 1, \ldots, p_1)$, and e_i is the number of rows of B_{ε_1} which are equal to $\gamma(1)^i$.

Proof. Let $[B_{\varepsilon_1}] \in M$, $B_{\varepsilon_1} = (b_{ij})$. We define $\Psi([B_{\varepsilon_1}]) = HgK$ with g given by:
For $l = (i-1) l_1 + b_{i1} + \cdots + b_{i(j-1)} + t$ $(1 \le t \le b_{ij})$

$$lg = a_{11} + \cdots + a_{1(j-1)} + b_{1j} + \cdots + b_{(i-1)j} + t. \tag{2}$$

If $[\bar{B}_{\varepsilon_1}] = [B_{\varepsilon_1}]$ with $\bar{B}_{\varepsilon_1} = (\bar{b}_{ij})$, let $\sigma \in \Sigma_{\varepsilon_1}$ be such that

$$\bar{b}_{i\sigma, j} = b_{ij} \quad (1 \le i \le \varepsilon_1;\ 1 \le j \le s).$$

Also, we put $\sigma = (1, \ldots, 1; \sigma) \in \Sigma_{\varepsilon_1} \langle \Sigma_{l_1} \rangle$.

Let l be as before and $\bar{g}$ defined for $\bar{B}_{\varepsilon_1}$ as g for B_{ε_1}. Now,

$$l\sigma = (i\sigma-1)l_1 + \bar{b}_{i\sigma,1} + \cdots + \bar{b}_{i\sigma,j-1} + t \quad (1 \le t \le \bar{b}_{i\sigma,j}),$$
$$(l\sigma)\bar{g} = a_{11} + \cdots + a_{1,j-1} + \bar{b}_{1j} + \cdots + \bar{b}_{i\sigma-1,j} + t.$$

So, $\sigma\bar{g} = g\tau$ for some $\tau \in K$. Then Ψ is well defined.

We define $\Phi : C \to M$ by $\Phi(HgK) = [B_{\varepsilon_1}]$ with $B_{\varepsilon_1} = (b_{ij})$ and b_{ij} the cardinal of the intersection

$$\{(i-1)l_1+1, \ldots, il_1\}\, g \cap \{a_{11} + \cdots + a_{1,j-1}+1, \ldots, a_{11} + \cdots + a_{1j}\}.$$

Φ is well defined; in fact:

Let $\bar{g} = hgk$ $(h \in H,\ k \in K)$ and suppose $h = (x_1, \ldots, x_{\varepsilon_1}; \sigma)$. Then $\bar{g}$ yields the matrix $\bar{B}_{\varepsilon_1} = (\bar{b}_{ij})$ with $\bar{b}_{ij}$ the cardinal of each intersection:

$$\{(i-1)l_1+1, \ldots, il_1\}\, hgk \cap \{a_{11} + \cdots + a_{1,j-1}+1, \ldots, a_{11} + \cdots + a_{1j}\},$$
$$\{(i-1)l_1+1, \ldots, il_1\}\, \sigma g \cap \{a_{11} + \cdots + a_{1,j-1}+1, \ldots, a_{11} + \cdots + a_{1j}\}k^{-1},$$
$$\{(i\sigma-1)l_1+1, \ldots, (i\sigma)l_1\}\, g \cap \{a_{11} + \cdots + a_{1,j-1}+1, \ldots, a_{11} + \cdots + a_{1j}\}.$$

So, $\bar{b}_{ij} = b_{i\sigma,j}$ and then $[B_{\varepsilon_1}] = [\bar{B}_{\varepsilon_1}]$.

Clearly, $\Phi\Psi([B_{\varepsilon_1}]) = [B_{\varepsilon_1}]$.

When $\Phi(HgK) = [B_{\varepsilon_1}]$ we may choose the representative g acting exactly as in (2) and then it is clear that

$$\Psi\Phi(HgK) = HgK.$$

Finally, suppose $\Psi([B_{\varepsilon_1}]) = HgK$. Let us choose B_{ε_1} such that the first e_1 rows are equal to $\gamma(1)^1$, the following e_2 rows are $\gamma(1)^2$, etc. The representative g acts as in (2). Then, $K^{g^{-1}}$ is the subgroup of $\Sigma_{\varepsilon_1 l_1}$ of all permutations fixing the sets X_r for $1 \le r \le s$ where j is in X_r if and only if

$$1 + kl_1 + \sum_{i=1}^{r-1} b_{k+1,i} \le j \le kl_1 + \sum_{j=1}^{r} b_{k+1,i}$$

for some k with $0 \le k < \varepsilon_1$.

The elements in H are those of $\Sigma_{l_1\varepsilon_1}$ which permute the sets $\{1, 2, \ldots, l_1\}$, $\{l_1+1, \ldots, 2l_1\}, \ldots, \{(\varepsilon_1-1)l_1+1, \ldots, \varepsilon_1 l_1\}$.

It is straightforward but tedious to check that $H \cap K^{g^{-1}}$ coincides with the subgroup of $\Sigma_{\varepsilon_1 l_1}$ given above. $\square$

Lemma 4.10. *Let $\Gamma_1, \ldots, \Gamma_s$ be finite groups, $d_i \in B(\Gamma_i)$ for $i = 1, \ldots, s$ and $e \in \mathbb{N}$. Let $\bar{\phi}$ be the group homomorphism*

$$\Sigma_e\langle \Gamma_1 \times \cdots \times \Gamma_s \rangle \to \Sigma_e\langle \Gamma_1 \rangle \times \cdots \times \Sigma_e\langle \Gamma_s \rangle.$$

Then, $\bar{\phi}^*(\bigotimes_{i=1}^{s} T_e(d_i)) = T_e(\bigotimes_{i=1}^{s} d_i)$.

Proof. First we establish some notations:

$G=\Sigma_e\langle\Gamma_1\rangle\times\cdots\times\Sigma_e\langle\Gamma_s\rangle$.

$K_j=\Sigma_j\langle\Gamma_2\times\cdots\times\Gamma_s\rangle$, $j=0,1,\dots,e$.

$\bar{K}_j=\Sigma_j\langle\Gamma_1\times\cdots\times\Gamma_s\rangle$, $j=0,1,\dots,e$.

$H_j=(\Sigma_j\langle\Gamma_1\rangle\times\Sigma_{e-j}\langle\Gamma_1\rangle)\times\Sigma_e\langle\Gamma_2\rangle\times\cdots\times\Sigma_e\langle\Gamma_s\rangle$, $j=0,1,\dots,s$.

In the following diagram all homomorphisms behave as expected:

$$\begin{tikzcd}
 & & G & & \\
H_j \arrow[urr, "\psi"] \arrow[r] & \psi(H_j) \arrow[ur, hook] & & \bar{\phi}(\bar{K}_e) \arrow[ul, hook'] & \bar{K}_e \arrow[l] \arrow[ull, "\bar{\phi}"'] \\
(\Sigma_j\langle\Gamma_1\rangle\times\Sigma_{e-j}\langle\Gamma_1\rangle)\times K_e \arrow[u, "\phi"] & & \psi(H_j)\cap\bar{\phi}(\bar{K}_e) \arrow[ul, hook] \arrow[ur, hook'] & & \\
(\Sigma_j\langle\Gamma_1\rangle\times\Sigma_{e-j}\langle\Gamma_1\rangle)\times K_j\times K_{e-j} \arrow[u, "\alpha"] & & & & \\
(\Sigma_j\langle\Gamma_1\rangle\times K_j)\times(\Sigma_{e-j}\langle\Gamma_1\rangle\times K_{e-j}) \arrow[u, "\tau"] & & & & \\
 & \bar{K}_j\times\bar{K}_{e-j} \arrow[ul, "\delta"] \arrow[rrr, "\gamma"] & & & (\Sigma_j\times\Sigma_{e-j})\langle\Gamma_1\times\cdots\times\Gamma_s\rangle \arrow[uuull] \arrow[uuuu, "\beta"']
\end{tikzcd}$$

Observe that G has only one $(\psi(H_j),\ \bar{\phi}(\bar{K}_e))$-double coset.

Now, applying the Mackey theorem and using the commutativity of the diagram and Proposition (3.3)(iv), one has

$$\bar{\phi}^*\psi_*=\beta_*\gamma_*\delta^*\tau^*\alpha^*\phi^* \tag{3}$$

Now, let us prove the lemma.

If $d_1=X_i$ is a Γ_i-set for all $i=1,\dots,s$, then it is easy to check the result. We will prove it for $d_1=X_1-X_1'$, $d_2=X_2,\dots,d_s=X_s$ with X_i a Γ_i-set $(i=1,\dots,s)$ and X_1' a Γ_1-set. With a similar process we would do it after that for any $d_2\in B(\Gamma_2)$, and so on.

Put $z:=\bar{\phi}^*[T_e(d_1)\otimes T_e(X_2)\otimes\cdots\otimes T_e(X_s)]$.

Now,

$$\begin{aligned}
&\bar{\phi}^*[T_e(X_1)\otimes T_e(X_2)\otimes\cdots\otimes T_e(X_s)]\\
&\quad=\sum_{j=0}^{e}\bar{\phi}^*[(T_j(d_1)\underline{\times}T_{e-j}(X_1'))\otimes T_e(X_2)\otimes\cdots\otimes T_e(X_s)]
\end{aligned}$$

$$= z + \sum_{j=0}^{e-1} \bar{\varphi}^* \psi_* [(T_j(d_1) \otimes T_{e-j}(X_1')) \otimes T_e(X_2) \otimes \cdots \otimes T_e(X_s)]$$

by (3) and the inductive hypothesis on s

$$= z + \sum_{j=0}^{e-1} \beta_* \gamma_* \delta^* \tau^* \alpha^* [(T_j(d_1) \otimes T_{e-j}(X_1')) \otimes T_e(X_2 \otimes \cdots \otimes X_s)]$$

by Proposition 4.2(iii)

$$= z + \sum_{j=0}^{e-1} \beta_* \gamma_* \delta^* \tau^* [(T_j(d_1) \otimes T_{e-j}(X_1')) \otimes (T_j(X_2 \otimes \cdots \otimes X_s) \otimes T_{e-j}(X_2 \otimes \cdots \otimes X_s))]$$

$$= z + \sum_{j=0}^{e-1} \beta_* \gamma_* \delta^* [(T_j(d_1) \otimes T_j(X_2 \otimes \cdots \otimes X_s)) \otimes (T_{e-j}(X_1') \otimes T_{e-j}(X_2 \otimes \cdots \otimes X_s))]$$

by the inductive hypothesis on e for $s=2$

$$= z + \sum_{j=0}^{e-1} \beta_* \gamma_* [T_j(d_1 \otimes X_2 \otimes \cdots \otimes X_s) \otimes T_{e-j}(X_1' \otimes X_2 \otimes \cdots \otimes X_s)]$$

by the definition of the cross product

$$= z + \sum_{j=0}^{e-1} [T_j(d_1 \times X_2 \times \cdots \times X_s) \underline{\times} T_{e-j}(X_1' \times X_2 \times \cdots \times X_s)].$$

On the other hand,

$$\bar{\varphi}^*(T_e(X_1) \otimes \cdots \otimes T_e(X_s)) = T_e(X_1 \otimes \cdots \otimes X_s)$$
$$= \sum_{j=0}^{e} [T_j(d_1 \otimes X_2 \otimes \cdots \otimes X_s) \underline{\times} T_{e-j}(X_1' \otimes X_2 \otimes \cdots \otimes X_s)].$$

Thus,

$$z = T_e(d_1 \otimes X_2 \otimes \cdots \otimes X_s). \qquad \square$$

Notations for the remainder of Section 4 (*except Lemmas* 4.15 *and* 4.18)

$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ is an r-partition of k.
$k(\varepsilon) := \varepsilon_1 l_1 + \cdots + \varepsilon_r l_r$.
$\omega(\varepsilon) = (\omega(\varepsilon)_1, \ldots, \omega(\varepsilon)_s)$ is an s-partition of $k(\varepsilon)$.
$\theta_\varepsilon' : \Sigma_{\varepsilon_1}\langle \Sigma_{l_1} \rangle \times \cdots \times \Sigma_{\varepsilon_r}\langle \Sigma_{l_r} \rangle \to \Sigma_{\varepsilon_1 l_1} \times \cdots \times \Sigma_{\varepsilon_r l_r}$.
$\theta_\varepsilon'' : \Sigma_{\varepsilon_1 l_1} \times \cdots \times \Sigma_{\varepsilon_r l_r} \to \Sigma_{k(\varepsilon)}$.
$\theta_\varepsilon = \theta_\varepsilon'' \theta_\varepsilon'$.
$\pi_\varepsilon : \Sigma_{\varepsilon_1}\langle \Sigma_{l_1} \rangle \times \cdots \times \Sigma_{\varepsilon_r}\langle \Sigma_{l_r} \rangle \to \Sigma_{\varepsilon_1} \times \cdots \times \Sigma_{\varepsilon_r}$.
$\delta_\varepsilon : \Sigma_{\varepsilon_1} \times \cdots \times \Sigma_{\varepsilon_r} \to \Sigma_k$.
$L(\varepsilon, b, a) = (\bigotimes_{j=1}^r T_{\varepsilon_j}(a_j))(\pi_\varepsilon^* \delta_\varepsilon^*(b))$.
$\theta_{\omega(\varepsilon)}' : \Sigma_{\omega(\varepsilon)_1}\langle \Sigma_{m_1} \rangle \times \cdots \times \Sigma_{\omega(\varepsilon)_s}\langle \Sigma_{m_s} \rangle \to \Sigma_{\omega(\varepsilon)_1 m_1} \times \cdots \times \Sigma_{\omega(\varepsilon)_s m_s}$.
$\theta_{\omega(\varepsilon)}'' : \Sigma_{\omega(\varepsilon)_1 m_1} \times \cdots \times \Sigma_{\omega(\varepsilon)_s m_s} \to \Sigma_{\omega(\varepsilon)_1 m_1 + \cdots + \omega(\varepsilon)_s m_s}$.

$\theta_{\omega(\varepsilon)}=\theta''_{\omega(\varepsilon)}\theta'_{\omega(\varepsilon)}$.
$\pi_{\omega(\varepsilon)}:\Sigma_{\omega(\varepsilon)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\omega(\varepsilon)_s}\langle\Sigma_{m_s}\rangle\to\Sigma_{\omega(\varepsilon)_1}\times\cdots\times\Sigma_{\omega(\varepsilon)_s}$.
$\delta_{\omega(\varepsilon)}:\Sigma_{\omega(\varepsilon)_1}\times\cdots\times\Sigma_{\omega(\varepsilon)_s}\to\Sigma_{k(\varepsilon)}$.
$\{\gamma(j)^1,\ldots,\gamma(j)^{p_j}\}$ is the set of all s-partitions of l_j.
$\gamma(j)^i=(\gamma(j)^i_1,\ldots,\gamma(j)^i_s)$.
$t:=p_1+\cdots+p_r$.
$A=(a_{ij})$ is an $\mathbb{N}$-matrix $r\times s$ with row sums $(\varepsilon_1 l_1,\ldots,\varepsilon_r l_r)$ and column sums $(\omega(\varepsilon)_1,\ldots,\omega(\varepsilon)_s)$.
B_{ε_j} is an $\mathbb{N}$-matrix $\varepsilon_j\times s$ with row sums $(l_j,\ldots,l_j)$ and column sums $(a_{j1},\ldots,a_{js})$. The classes $[B_{\varepsilon_j}]$ are defined as in Lemma 4.9.
B is the 'diagonal' sum of $[B_{\varepsilon_j}]$ $(j=1,\ldots,r)$.

$$B=\begin{array}{c}\left[\begin{array}{ccc}[B_{\varepsilon_1}] & & 0\\ & [B_{\varepsilon_2}] & \\ 0 & & \ddots\end{array}\right]\begin{array}{l}\}\,\varepsilon_1\\ \}\,\varepsilon_2\\ \vdots\end{array}\\ \underbrace{\quad}_{s}\ \underbrace{\quad}_{s}\ \cdots\end{array}$$

$\mu_A:(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{r1}})\times\cdots\times(\Sigma_{a_{1s}}\times\cdots\times\Sigma_{a_{rs}})$
$\quad\to(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{1s}})\times\cdots\times(\Sigma_{a_{r1}}\times\cdots\times\Sigma_{a_{rs}})$.
$\varrho_A:(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{1s}})\times\cdots\times(\Sigma_{a_{r1}}\times\cdots\times\Sigma_{a_{rs}})\to\Sigma_{\varepsilon_1 l_1}\times\cdots\times\Sigma_{\varepsilon_r l_r}$.
$\phi_A:(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{r1}})\times\cdots\times(\Sigma_{a_{1s}}\times\cdots\times\Sigma_{a_{rs}})\to\Sigma_{\omega(\varepsilon)_1}\times\cdots\times\Sigma_{\omega(\varepsilon)_s}$.
$e_{p_1+\cdots+p_{j-1}+q}$ $(1\le q\le p_j)$ is the number of rows of B_{ε_j} which are equal to $\gamma(j)^q$.
$H_B=(\Sigma_{e_1}\langle\Sigma_{\gamma(1)^1_1}\times\cdots\times\Sigma_{\gamma(1)^1_s}\rangle\times\cdots\times\Sigma_{e_{p_1}}\langle\Sigma_{\gamma(1)^{p_1}_1}\times\cdots\times\Sigma_{\gamma(1)^{p_1}_s}\rangle)\times\cdots$
$\quad\cdots\times(\Sigma_{e_{t-p_r+1}}\langle\Sigma_{\gamma(r)^1_1}\times\cdots\times\Sigma_{\gamma(r)^1_s}\rangle\times\cdots\times\Sigma_{e_t}\langle\Sigma_{\gamma(r)^{p_r}_1}\times\cdots\times\Sigma_{\gamma(r)^{p_r}_s}\rangle)$.
$H_B\langle\Sigma_{m_i}\rangle$: put $\Sigma_{\gamma(j)^q_i}\langle\Sigma_{m_i}\rangle$ instead of $\Sigma_{\gamma(j)^q_i}$ in H_B for $j=1,\ldots,r$; $i=1,\ldots,s$; $q=1,\ldots,p_j$.
$\eta_B:H_B\to\Sigma_{\varepsilon_1}\langle\Sigma_{l_1}\rangle\times\cdots\times\Sigma_{\varepsilon_r}\langle\Sigma_{l_r}\rangle$.
$\tau:H^g_B\to H_B$ is conjugation by g^{-1}, where $g=g_1\times\cdots\times g_r$ and g_j is a representative of $\Psi([B_{\varepsilon_j}])$ in Lemma 4.9 and defined in the same way.
$\upsilon_B:H^g_B\hookrightarrow(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{1s}})\times\cdots\times(\Sigma_{a_{r1}}\times\cdots\times\Sigma_{a_{rs}})$.
$H^g_B\langle\Sigma_{m_i}\rangle$ is the subgroup of
$$\begin{aligned}&[(\Sigma_{e_1}\langle\Sigma_{\gamma(1)^1_1}\rangle\times\cdots\times\Sigma_{e_{p_1}}\langle\Sigma_{\gamma(1)^{p_1}_1}\rangle)\langle\Sigma_{m_1}\rangle\times\cdots\\ &\quad\cdots\times(\Sigma_{e_1}\langle\Sigma_{\gamma(1)^1_s}\rangle\times\cdots\times\Sigma_{e_{p_1}}\langle\Sigma_{\gamma(1)^{p_1}_s}\rangle)\langle\Sigma_{m_s}\rangle]\times\cdots\\ &\quad\cdots\times[(\Sigma_{e_{t-p_r+1}}\langle\Sigma_{\gamma(r)^1_1}\rangle\times\cdots\times\Sigma_{e_t}\langle\Sigma_{\gamma(r)^{p_r}_1}\rangle)\langle\Sigma_{m_1}\rangle\times\cdots\\ &\quad\cdots\times(\Sigma_{e_{t-p_r+1}}\langle\Sigma_{\gamma(r)^1_s}\rangle\times\cdots\times\Sigma_{e_t}\langle\Sigma_{\gamma(r)^{p_r}_s}\rangle)\langle\Sigma_{m_s}\rangle]\end{aligned}$$
consisting of all elements such that the 'component' in Σ_{e_i} is the same for all places where it occurs $(i=1,\ldots,t)$.
$\bar\phi_A:(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{r1}})\langle\Sigma_{m_1}\rangle\times\cdots\times(\Sigma_{a_{1s}}\times\cdots\times\Sigma_{a_{rs}})\langle\Sigma_{m_s}\rangle$
$\quad\to\Sigma_{\omega(\varepsilon)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\omega(\varepsilon)_s}\langle\Sigma_{m_s}\rangle$.
$\bar\pi_{\omega(\varepsilon)}:(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{r1}})\langle\Sigma_{m_1}\rangle\times\cdots\times(\Sigma_{a_{1s}}\times\cdots\times\Sigma_{a_{rs}})\langle\Sigma_{m_s}\rangle$
$\quad\to(\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{r1}})\times\cdots\times(\Sigma_{a_{1s}}\times\cdots\times\Sigma_{a_{rs}})$.

$\bar{\upsilon}_B : H_B^g\langle\Sigma_{m_i}\rangle \hookrightarrow (\Sigma_{a_{11}}\langle\Sigma_{m_1}\rangle \times\cdots\times \Sigma_{a_{1s}}\langle\Sigma_{m_s}\rangle)\times\cdots\times(\Sigma_{a_{r1}}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{a_{rs}}\langle\Sigma_{m_s}\rangle)$.
$\bar{\mu}_A : (\Sigma_{a_{11}}\times\cdots\times\Sigma_{a_{r1}})\langle\Sigma_{m_1}\rangle\times\cdots\times(\Sigma_{a_{1s}}\times\cdots\times\Sigma_{a_{rs}})\langle\Sigma_{m_s}\rangle$
$\quad\to(\Sigma_{a_{11}}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{a_{1s}}\langle\Sigma_{m_s}\rangle)\times\cdots\times(\Sigma_{a_{r1}}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{a_{rs}}\langle\Sigma_{m_s}\rangle)$.
$\pi_B : H_B\langle\Sigma_{m_i}\rangle\to H_B$.
$\bar{\pi}_B : H_B^g\langle\Sigma_{m_i}\rangle\to H_B^g$.
$\sigma=(\sigma_1,\ldots,\sigma_t)$ is a t-partition of k.
$\gamma(h)=\gamma(j)^q$ if $h=p_1+\cdots+p_{j-1}+q$ with $1\le q\le p_j$.
$n_h=\gamma(h)_1m_1+\cdots+\gamma(h)_sm_s$.
$\theta'_{\gamma(h)} : \Sigma_{\gamma(h)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\gamma(h)_s}\langle\Sigma_{m_s}\rangle\to\Sigma_{\gamma(h)_1m_1}\times\cdots\times\Sigma_{\gamma(h)_sm_s}$.
$\theta''_{\gamma(h)} : \Sigma_{\gamma(h)_1m_1}\times\cdots\times\Sigma_{\gamma(h)_sm_s}\to\Sigma_{n_h}$.
$\theta_{\gamma(h)}=\theta''_{\gamma(h)}\theta_{\gamma(h)}$.
$\delta_{\gamma(h)} : \Sigma_{\gamma(h)_1}\times\cdots\times\Sigma_{\gamma(h)_s}\to\Sigma_{l_j}$ if $\gamma(h)=\gamma(j)^q$.
$\pi_{\gamma(h)} : \Sigma_{\gamma(h)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\gamma(h)_s}\langle\Sigma_{m_s}\rangle\to\Sigma_{\gamma(h)_1}\times\cdots\times\Sigma_{\gamma(h)_s}$.
$L(\gamma(h),a_j,c)=(\bigotimes_{i=1}^s T_{\gamma(h)_i}(c_i))(\pi^*_{\gamma(h)}\delta^*_{\gamma(h)}(a_j))$.
$\theta'_\sigma : \Sigma_{\sigma_1}\langle\Sigma_{n_1}\rangle\times\cdots\times\Sigma_{\sigma_t}\langle\Sigma_{n_t}\rangle\to\Sigma_{\sigma_1n_1}\times\cdots\times\Sigma_{\sigma_tn_t}$.
$\theta''_\sigma : \Sigma_{\sigma_1n_1}\times\cdots\times\Sigma_{\sigma_tn_t}\to\Sigma_{\sigma_1n_1+\cdots+\sigma_tn_t}$.
$\theta_\sigma=\theta''_\sigma\theta'_\sigma$.
$\pi_\sigma : \Sigma_{\sigma_1}\langle\Sigma_{n_1}\rangle\times\cdots\times\Sigma_{\sigma_t}\langle\Sigma_{n_t}\rangle\to\Sigma_{\sigma_1}\times\cdots\times\Sigma_{\sigma_t}$.
$\delta_\sigma : \Sigma_{\sigma_1}\times\cdots\times\Sigma_{\sigma_t}\to\Sigma_k$.
$\bar{\theta}_{\gamma(h)} : \Sigma_{\sigma_h}\langle(\Sigma_{\gamma(h)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\gamma(h)_s}\langle\Sigma_{m_s}\rangle)\rangle\to\Sigma_{\sigma_h}\langle\Sigma_{n_h}\rangle$.
$\bar{\delta}_{\gamma(h)} : \Sigma_{\sigma_h}\langle\Sigma_{\gamma(h)_1}\times\cdots\times\Sigma_{\gamma(h)_s}\rangle\to\Sigma_{\sigma_h}\langle\Sigma_{l_j}\rangle$.
$\bar{\pi}_{\gamma(h)} : \Sigma_{\sigma_h}\langle(\Sigma_{\gamma(h)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\gamma(h)_s}\langle\Sigma_{m_s}\rangle)\rangle\to\Sigma_{\sigma_h}\langle\Sigma_{\gamma(h)_1}\times\cdots\times\Sigma_{\gamma(h)_s}\rangle$.
$\bar{\tau} : H_B^g\langle\Sigma_{m_i}\rangle\to H_B\langle\Sigma_{m_i}\rangle$ defined from τ in the obvious way.
$\psi_h^1 : \Sigma_{\sigma_h}\langle(\Sigma_{\gamma(h)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\gamma(h)_s}\langle\Sigma_{m_s}\rangle)\rangle\to\Sigma_{\sigma_h}\langle\Sigma_{\gamma(h)_1}\langle\Sigma_{m_1}\rangle\rangle\times\cdots\times\Sigma_{\sigma_h}\langle\Sigma_{\gamma(h)_s}\langle\Sigma_{m_s}\rangle\rangle$.
$\psi_h^2 : (\mathrm{codomain}(\psi_h^1))\to\Sigma_{\sigma_h\gamma(h)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\sigma_h\gamma(h)_s}\langle\Sigma_{m_s}\rangle$.
$\psi^3 : \times_{h=1}^t\ [\Sigma_{\sigma_h\gamma(h)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\sigma_h\gamma(h)_s}\langle\Sigma_{m_s}\rangle]$
$\quad\to[\Sigma_{\sigma_1\gamma(1)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\sigma_t\gamma(t)_1}\langle\Sigma_{m_1}\rangle]\times\cdots\times[\Sigma_{\sigma_1\gamma(1)_s}\langle\Sigma_{m_s}\rangle\times\cdots\times\Sigma_{\sigma_t\gamma(t)_s}\langle\Sigma_{m_s}\rangle]$.
$\psi^4 : (\mathrm{codomain}(\psi^3))\to\Sigma_{\omega(\varepsilon)_1}\langle\Sigma_{m_1}\rangle\times\cdots\times\Sigma_{\omega(\varepsilon)_s}\langle\Sigma_{m_s}\rangle$.
$\eta'_j=\bigotimes_{h_j}\bar{\delta}_{\gamma(h_j)}$ where $h_j=p_1+\cdots+p_{j-1}+1,\ldots,p_1+\cdots+p_j$ for $j=1,\ldots,r$.
$\eta''_j : \Sigma_{\sigma_{p_1+\cdots+p_{j-1}+1}}\langle\Sigma_{l_j}\rangle\times\cdots\times\Sigma_{\sigma_{p_1+\cdots+p_j}}\langle\Sigma_{l_j}\rangle\to\Sigma_{\varepsilon_j}\langle\Sigma_{l_j}\rangle$.

Proposition 4.11. *Property* 3 *of Definition* 4.4 *is valid in* $B(S)$.

Proof. Let $b\in B(\Sigma_k)$, $a=a_1+\cdots+a_r$ with $a_j\in B(\Sigma_{l_j})$ $(j=1,\ldots,r)$, $c=c_1+\cdots+c_s$ with $c_i\in B(\Sigma_{m_i})$ $(i=1,\ldots,s)$.

We want to prove $(b\vee a)\vee c=b\vee(a\vee c)$.

$$(b\vee a)\vee c=\left[\sum_\varepsilon(\theta_\varepsilon)_*L(\varepsilon,b,a)\right]\vee c$$

$$=\sum_\varepsilon\sum_{\omega(\varepsilon)}(\theta_{\omega(\varepsilon)})_*\left[\left(\bigotimes_{i=1}^s T_{\omega(\varepsilon)_i}(c_i)\right)(\pi^*_{\omega(\varepsilon)}\delta^*_{\omega(\varepsilon)}(\theta_\varepsilon)_*L(\varepsilon,b,a))\right]$$

by the Mackey theorem and Lemma 4.7 (put $U=(\bigotimes_{i=1}^s T_{\omega(\varepsilon)_i}(c_i))$)

$$= \sum_{\varepsilon} \sum_{\omega(\varepsilon)} (\theta_{\omega(\varepsilon)})_* \left[U\left(\pi^*_{\omega(\varepsilon)} \left[\sum_A (\phi_A)_* \mu_A^* \varrho_A^* (\theta'_\varepsilon)_* L(\varepsilon, b, a) \right] \right) \right]$$

by the Mackey theorem and Lemma 4.9

$$= \sum_{\varepsilon} \sum_{\omega(\varepsilon)} \sum_A (\theta_{\omega(\varepsilon)})_* \left[U\left(\pi^*_{\omega(\varepsilon)} (\phi_A)_* \mu_A^* \left[\sum_B (\upsilon_B)_* \tau^* \eta_B^* L(\varepsilon, b, a) \right] \right) \right] =$$

by Proposition 3.6: $\pi^*_{\omega(\varepsilon)}(\phi_A)_* = (\bar\phi_A)_* \bar\pi^*_{\omega(\varepsilon)}$; so, $\bar\pi^*_{\omega(\varepsilon)} \mu_A^* (\upsilon_B)_* = \bar\pi^*_{\omega(\varepsilon)} (\mu_A^{-1})_* (\upsilon_B)_* = \bar\pi^*_{\omega(\varepsilon)} (\mu_A^{-1} \upsilon_B)_* = (\bar\mu_A^{-1})_* (\bar\upsilon_B)_* \bar\pi_B^*$

$$= \sum_{\varepsilon} \sum_{\omega(\varepsilon)} \sum_A \sum_B (\theta_{\omega(\varepsilon)})_* [U((\bar\phi_A)_* (\bar\mu_A^{-1})_* (\bar\upsilon_B)_* \bar\pi_B^* \tau^* \eta_B^* L(\varepsilon, b, a))]$$

by Lemma 4.12 and Frobenius reciprocity

$$= \sum_{\sigma \in D} (\theta_{\omega(\varepsilon)} \bar\phi_A \bar\mu_A^{-1} \bar\upsilon_B)_* [[\bar\upsilon_B^* (\bar\mu_A^{-1})^* \bar\phi_A^* (U)] [\bar\pi_B^* \tau^* \eta_B^* L(\varepsilon, b, a)]]$$

by Lemma 4.13 and $\zeta_* = \mathrm{id}$

$$= \sum_{\sigma} \left(\theta_\sigma \left(\bigotimes_{h=1}^{t} \bar\theta_{\gamma(h)} \right) \bar\tau \right)_* [[\bar\upsilon_B^* (\bar\mu_A^{-1})^* \bar\phi_A^* (U)] [\bar\pi_B^* \tau^* \eta_B^* L(\varepsilon, b, a)]]$$

by Lemma 4.14

$$= \sum_{\sigma} \left(\theta_\sigma \left(\bigotimes_{h=1}^{t} \bar\theta_{\gamma(h)} \right) \bar\tau \right)_* \left[\left(\bar\tau^* \left[\bigotimes_{h=1}^{t} T_{\sigma_h} \left(\bigotimes_{i=1}^{s} T_{\gamma(h)_i}(c_i) \right) \right] \right) \right.$$
$$\left. (\bar\pi_B^* \tau^* \eta_B^* L(\varepsilon, b, a)) \right] =$$

by Proposition 3.3 (put $\bigotimes_{h=1}^t T_{\sigma_h}(\bigotimes_{i=1}^s T_{\gamma(h)_i}(c_i)) = V$ and $\theta_\sigma(\bigotimes_{h=1}^t \bar\theta_{\gamma(h)})\bar\tau = \Lambda$)

$$= \sum_{\sigma} \Lambda_* \left[\bar\tau^*(V) \left(\bar\pi_B^* \tau^* \eta_B^* \left[\bigotimes_{j=1}^{r} T_{\varepsilon_j}(a_j) \right] \right) (\bar\pi_B^* \tau^* \eta_B^* \pi_\varepsilon^* \delta^*(b)) \right]$$

by Lemma 4.16

$$= \sum_{\sigma} \Lambda_* \left[\bar\tau^*(V) \left(\bar\tau^* \left[\bigotimes_{h=1}^{t} T_{\sigma_h} (\pi^*_{\gamma(h)} \delta^*_{\gamma(h)}(a_h)) \right] \right) (\bar\pi_B^* \tau^* \eta_B^* \pi_\varepsilon^* \delta_\varepsilon^*(b)) \right]$$

by Lemma 4.17 and Proposition 3.3(i), (iv) applied to $\bar\tau$

$$= \sum_{\sigma} (\theta_\sigma)_* \left(\bigotimes_{h=1}^{t} \bar\theta_{\gamma(h)} \right)_* \left[V \left(\bigotimes_{h=1}^{t} T_{\sigma_h} (\pi^*_{\gamma(h)} \delta^*_{\gamma(h)}(a_h)) \right) \right.$$
$$\left. \left(\left(\bigotimes_{h=1}^{t} \bar\theta_{\gamma(h)} \right)^* \pi_\sigma^* \delta_\sigma^*(b) \right) \right]$$

by Proposition 4.2(iv) (remember $V = \cdots$) and Frobenius reciprocity

$$= \sum_{\sigma} (\theta_\sigma)_* \left[\left(\bigotimes_{h=1}^{t} (\bar{\theta}_{\gamma(h)})_* T_{\sigma_h}[L(\gamma(h), a_h, c)] \right) (\pi_\sigma^* \delta_\sigma^*(b)) \right]$$

by Lemma 4.18

$$= \sum_{\sigma} (\theta_\sigma)_* \left[\left(\bigotimes_{h=1}^{t} T_{\sigma_h}[(\theta_{\gamma(h)})_* L(\gamma(h), a_h, c)] \right) (\pi_\sigma^* \delta_\sigma^*(b)) \right]$$

$$= b \vee (a \vee c).$$

The last equality is due to

$$a \vee c = \sum_{j=1}^{r} a_j \vee c = \sum_{j=1}^{r} \left[\sum_{h_j} (\theta_{\gamma(h_j)})_* L(\gamma(h_j), a_j, c) \right]$$

$$= \sum_{h=1}^{t} (\theta_{\gamma(h)})_* L(\gamma(h), a_h, c)$$

where $h_j = p_1 + \cdots + p_{j-1} + 1, \ldots, p_1 + \cdots + p_j$ and $a_h = a_j$ when h is equal to some h_j. □

Lemma 4.12. *Let $D = \{\sigma = (\sigma_1, \ldots, \sigma_t)$: σ a t-partition of $k\}$ and $M = \{B$: B as defined in notations$\}$.*

Then there exists a bijective map $\chi : D \to M$.

Proof. For $\sigma \in D$ we define $\chi(\sigma) = B$, the 'diagonal sum' of $[B_{\varepsilon_j}]$ $(j = 1, \ldots, r)$ with B_{ε_j} the matrix consisting of σ_h rows equal to $\gamma(j)^q$, where $h = p_1 + \cdots + p_{j-1} + q$ and $1 \le q \le p_j$.

Mapping each matrix B to the partition σ which has σ_h = number of rows of the jth 'diagonal' component of B, (B_{ε_j}), equal to $\gamma(j)^q$ $(h = p_1 + \cdots + p_{j-1} + q)$ we obtain an inverse map of χ. □

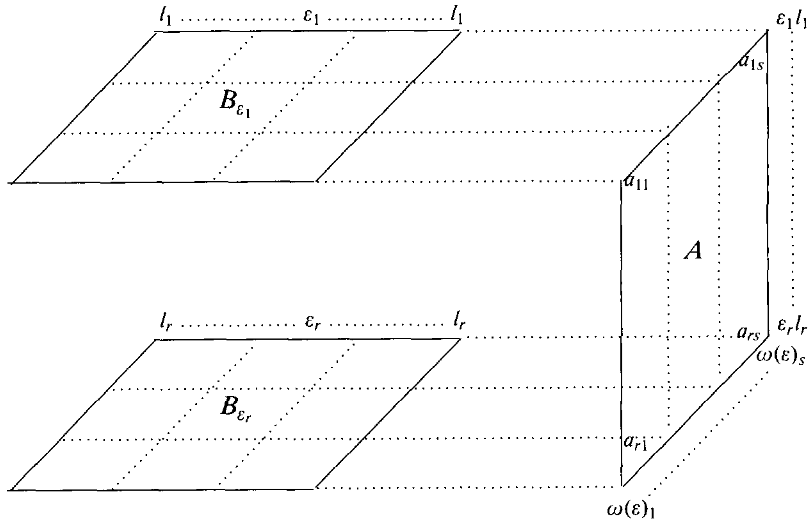
Observations.

$$\operatorname{card} D = \operatorname{card} M = \sum_{\varepsilon} \begin{bmatrix} p_1 \\ \varepsilon_1 \end{bmatrix} \cdots \begin{bmatrix} p_r \\ \varepsilon_r \end{bmatrix},$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ ranges over all r-partitions of k and $\binom{p_j}{\varepsilon_j}$ is the number of combinations of p_j elements, ε_j at a time, when repetitions are allowed.

Note that now σ determines an r-partition of k: $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ with $\varepsilon_j = \sigma_{p_1 + \cdots + p_{j-1} + 1} + \cdots + \sigma_{p_1 + \cdots + p_j}$, a matrix A whose rows are the column sums of the matrices B_{ε_j}, an s-partition of $k(\varepsilon)$: $\omega(\varepsilon)$ given by the column sums of A, and obviously a matrix B.

The following picture may be useful:



Lemma 4.13. *There exists an inner automorphism ζ of the group $\Sigma_{\omega(\varepsilon)_1 m_1+\cdots+\omega(\varepsilon)_s m_s}$ such that*

$$\zeta\theta_\sigma\left(\bigotimes_{h=1}^{t}\bar{\theta}_{\gamma(h)}\right)\bar{\tau}=\theta_{\omega(\varepsilon)}\bar{\phi}_A\bar{\mu}_A^{-1}\bar{v}_B.$$

Proof. First, observe that

$$\sigma_1 n_1+\cdots+\sigma_t n_t=\omega(\varepsilon)_1 m_1+\cdots+\omega(\varepsilon)_s m_s.$$

Now, forgetting ζ, both compositions are homomorphisms from $H_B^g\langle\Sigma_{m_i}\rangle$ to $\Sigma_{\sigma_1 n_1+\cdots+\sigma_t n_t}$ and all components are either inclusions or conjugations in $\Sigma_{\sigma_1 n_1+\cdots+\sigma_t n_t}$. $\square$

Lemma 4.14.

$$\bar{v}_B^*(\bar{\mu}_A^{-1})^*\bar{\phi}_A^*\left(\bigotimes_{i=1}^{s}T_{\omega(\varepsilon)_i}(c_i)\right)=\bar{\tau}^*\left[\bigotimes_{h=1}^{t}T_{\sigma_h}\left(\bigotimes_{i=1}^{s}T_{\gamma(h)_i}(c_i)\right)\right].$$

Proof. It is not difficult to check that

$$\bar{\phi}_A\mu_A^{-1}\bar{v}_B=\psi^4\psi^3\left(\bigotimes_{h=1}^{t}\psi_h^2\right)\left(\bigotimes_{h=1}^{t}\psi_h^1\right)\bar{\tau}.$$

Now,

$$\begin{aligned}&\bar{v}_B^*(\bar{\mu}_A^{-1})^*\bar{\phi}_A^*\left(\bigotimes_{i=1}^{s}T_{\omega(\varepsilon)_i}(c_i)\right)\\&=\bar{\tau}^*\left(\bigotimes_{h=1}^{t}\psi_h^1\right)^*\left(\bigotimes_{h=1}^{t}\psi_h^2\right)^*(\psi^3)^*(\psi^4)^*\left[\bigotimes_{i=1}^{s}T_{\omega(\varepsilon)_i}(c_i)\right]\end{aligned}$$

by Proposition 4.2(iii)

$$= \bar{\tau}^*\left(\bigotimes_{h=1}^{t} \psi_h^1\right)^*\left(\bigotimes_{h=1}^{t} \psi_h^2\right)^*(\psi^3)^*\left[\bigotimes_{i=1}^{s}\left(\bigotimes_{h=1}^{t} T_{\sigma_h \gamma(h)_i}(c_i)\right)\right]$$

$$= \bar{\tau}^*\left(\bigotimes_{h=1}^{t} \psi_h^1\right)^*\left(\bigotimes_{h=1}^{t} \psi_h^2\right)^*\left[\bigotimes_{h=1}^{t}\left(\bigotimes_{i=1}^{s} T_{\sigma_h \gamma(h)_i}(c_i)\right)\right]$$

by Proposition 4.2(v)

$$= \bar{\tau}^*\left[\bigotimes_{h=1}^{t} (\psi_h^1)^*\left(\bigotimes_{i=1}^{s} T_{\sigma_h} T_{\gamma(h)_i}(c_i)\right)\right]$$

by Lemma 4.10

$$= \bar{\tau}^*\left[\bigotimes_{h=1}^{t} T_{\sigma_h}\left(\bigotimes_{i=1}^{s} T_{\gamma(h)_i}(c_i)\right)\right]. \qquad \square$$

Lemma 4.15. *Let $\Psi: \Gamma \to \Omega$ be a group homomorphism and $\bar{\Psi}_k = \Sigma_k\langle\Psi\rangle : \Sigma_k\langle\Gamma\rangle \to \Sigma_k\langle\Omega\rangle$ $(k \in \mathbb{N})$. Then*

$$T_k \Psi^* = \bar{\Psi}^* T_k.$$

Proof. The following diagram commutes for Ω-sets:

$$\begin{tikzcd} B(\Omega) \arrow[r, "\Psi^*"] \arrow[d, "T"'] & B(\Gamma) \arrow[d, "T"] \\ \prod_{k\geq 0} B(\Sigma_k\langle\Omega\rangle) \arrow[r, "\prod_{k\geq 0} \bar{\Psi}_K^*"'] & \prod_{k\geq 0} B(\Sigma_k\langle\Gamma\rangle) \end{tikzcd}$$

$T\Psi^*$ is a homomorphism from '+' to '$\underline{\times}$'. Finally, $(\prod_{k\geq 0} \bar{\Psi}_k^*)T$ is a homomorphism from '+' to '$\underline{\times}$' because $\prod_{k\geq 0} \bar{\Psi}_k^*$ is from '$\underline{\times}$' to '$\underline{\times}$' since the following diagram commutes:

$$\begin{tikzcd} B(\Sigma_k\langle\Omega\rangle) \arrow[r, "\bar{\Psi}_K^*"] & B(\Sigma_k\langle\Gamma\rangle) \\ B(\Sigma_i\langle\Omega\rangle)\otimes B(\Sigma_j\langle\Omega\rangle) \arrow[u, "`\underline{\times}'"] \arrow[r, "\bar{\Psi}_i^*\otimes\bar{\Psi}_j^*"'] & B(\Sigma_i\langle\Gamma\rangle)\otimes B(\Sigma_j\langle\Gamma\rangle) \arrow[u, "`\underline{\times}'"'] \end{tikzcd}$$

Lemma 4.16.

$$\bar{\pi}_B^* \tau^* \eta_B^*\left(\bigotimes_{j=1}^{r} T_{\varepsilon_j}(a_j)\right) = \bar{\tau}^*\left[\bigotimes_{h=1}^{t} T_{\sigma_h}(\pi_{\gamma(h)}^* \delta_{\gamma(h)}^*(a_h))\right]$$

where $a_h = a_j$ if $h \in \{p_1 + \cdots + p_{j-1} + 1, \ldots, p_1 + \cdots + p_j\}$.

Proof. $\eta_B = (\bigotimes_{j=1}^r \eta_j'')(\bigotimes_{j=1}^r \eta_j')$. So,

$$\bar{\pi}_B^* \tau^* \eta_B^* \left(\bigotimes_{j=1}^r T_{\varepsilon_j}(a_j) \right) = \bar{\pi}_B^* \tau^* \left(\bigotimes_{j=1}^r \eta_j' \right)^* \left[\bigotimes_{j=1}^r (\eta_j'')^* T_{\varepsilon_j}(a_j) \right]$$

by Proposition 4.2(iii) and putting $h_j = p_1 + \cdots + p_{j-1} + 1, \ldots, p_1 + \cdots + p_j$

$$= \bar{\pi}_B^* \tau^* \left(\bigotimes_{j=1}^r \eta_j' \right)^* \left[\bigotimes_{j=1}^r \left(\bigotimes_{h_j} T_{\sigma_{h_j}}(a_j) \right) \right]$$

$$= \bar{\pi}_B^* \tau^* \left[\bigotimes_{j=1}^r \left(\bigotimes_{h_j} \bar{\delta}_{\gamma(h_j)}^* T_{\sigma_{h_j}}(a_j) \right) \right] = \bar{\pi}_B^* \tau^* \left[\bigotimes_{h=1}^t \bar{\delta}_{\gamma(h)}^* T_{\sigma_h}(a_h) \right]$$

by $\tau \bar{\pi}_B = \pi_B \bar{\tau}$

$$= \bar{\tau}^* \pi_B^* \left[\bigotimes_{h=1}^t \bar{\delta}_{\gamma(h)}^* T_{\sigma_h}(a_h) \right] = \bar{\tau}^* \left[\bigotimes_{h=1}^t \pi_{\gamma(h)}^* \bar{\delta}_{\gamma(h)}^* T_{\sigma_h}(a_h) \right]$$

by Lemma 4.15

$$= \bar{\tau}^* \left[\bigotimes_{h=1}^t T_{\sigma_h}(\pi_{\gamma(h)}^* \delta_{\gamma(h)}^*(a_h)) \right]. \qquad \square$$

Lemma 4.17. $\delta_\varepsilon \pi_\varepsilon \eta_B \tau \bar{\pi}_B = \delta_\sigma \pi_\sigma (\bigotimes_{h=1}^t \bar{\theta}_{\gamma(h)}) \bar{\tau}$.

Proof. $\tau \bar{\pi}_B = \pi_B \bar{\tau}$, and then it is easy to check that

$$\delta_\varepsilon \pi_\varepsilon \eta_B \pi_B = \delta_\sigma \pi_\sigma \left(\bigotimes_{h=1}^t \bar{\theta}_{\gamma(h)} \right). \qquad \square$$

Lemma 4.18. *Let $\Psi : \Omega \to \Gamma$ be a monomorphism of finite groups. Let $\bar{\Psi}_k = \Sigma_k\langle\Psi\rangle : \Sigma_k\langle\Omega\rangle \to \Sigma_k\langle\Gamma\rangle$ ($k \in \mathbb{N}$). Then*

$$T_k \Psi_* = (\bar{\Psi}_k)_* T_k.$$

Proof. Similarly to Lemma 4.15, it is easy to show that both compositions are homomorphisms from '$+$' to '$\underline{\times}$'. Then, it is enough to check the commutativity for Ω-sets, and this can be checked as in [9, Proposition 2.9]. $\square$

Properties 1 and 4 of Definition 4.4 are valid in $B(S)$ (take for 'e' the Σ_1-set of one element). Then, from Propositions 4.8 and 4.11, we have

Theorem 4.19. *The operation composition '$\vee$' defined in Definition 4.5 makes the graded ring $B(S) = \bigoplus_{k \geq 0} B(\Sigma_k)$ an operator ring.* $\square$

Note. Rymer [12] states the same theorem by using the definition of Burnside ring of a compact Lie group to avoid formal negatives in constructing a group from a semigroup. But he defines the operation $(b \vee a)$ only for elements 'a' in $B(\Sigma_k)$ (any k) and not for any element 'a' in $B(S)$.

5. Some notes on β-rings

First we take the definition of β-ring given by Rymer in [12]:

Definition 5.1. A *β-ring* is a commutative ring C with identity and a composition $B(S) \times C \to C$ satisfying:

(1) $(a+b) \cdot c = (a \cdot c) + (b \cdot c)$,

(2) $(a \underline{\times} b) \cdot c = (a \cdot c)(b \cdot c)$,

(3) $(a \vee b) \cdot c = a \cdot (b \cdot c)$,

for all a, $b \in B(S)$, $c \in C$.

Let Γ be a finite group. We define the composition

$$B(\Sigma_r) \times B(\Gamma) \to B(\Gamma)$$

by means of

$$a \cdot c = (\eta_r)_* \alpha_r^* [\pi_r^*(a) T_r(c)]$$

where

$$a \in B(\Sigma_r), \qquad c \in B(\Gamma),$$

$$\alpha_r : \Sigma_r \times \Gamma \to \Sigma_r \langle \Gamma \rangle \quad ((\sigma, g) \mapsto (g, \dots, g; \sigma)),$$

$$\pi_r : \Sigma_r \langle \Gamma \rangle \to \Sigma_r, \qquad \eta_r : \Sigma_r \times \Gamma \to \Gamma.$$

This composition can be extended by left additivity to all elements in $B(S)$. But it does not seem to be easy to prove that $B(\Gamma)$ becomes a β-ring. In particular, property 3 may be complicated, because it is not enough to take $b \in B(\Sigma_m)$ for some m; in fact, we have to take $b = b_1 + \cdots + b_s$, with $b_j \in B(\Sigma_{m_j})$ $(j = 1, \dots, s)$.

Theorem 5.2. *The homomorphism $\Psi : B(S) \to R(S)$ induced by assigning to a finite Σ_r-set X the complex Σ_r-representation $\mathbb{C}(X)$ on the set X as basis is a homomorphism of operator rings.*

Proof. The theorem follows from the definitions of '$\vee$' in $B(S)$ and $R(S)$ by showing that Ψ commutes with inductions, restrictions and with the maps T_k ($\bar{\tau}_k$ of [9]). $\square$

Note that it is not enough to prove $\Psi(b \vee a) = \Psi(b) \vee \Psi(a)$ with $a \in B(\Sigma_r)$ for some $r \in \mathbb{N}$, as Rymer does. In fact, it is not possible to apply [5, Lemma (4.6)] because the set $\{a \in B(\Sigma_r);\ r \in \mathbb{N}\}$ is not closed under addition.

Let Λ be the ring of all λ-operations. By composing $\theta\Psi$, where $\theta : R(S) \to \Lambda$ is the map $a \mapsto \tau^a$ (see [9]), we obtain a homomorphism of operator rings $B(S) \to \Lambda$. So, every λ-ring is a β-ring.

From now on we use the definition of β-ring given by Morris and Wensley [11]:

Let $P = \bigcup_{r\geq 0} \{\Sigma_r\text{-sets}\}$.

Definition 5.3. A *β-ring* is a commutative ring C with identity together with a function:

$$\beta : P \times C \to C, \qquad \beta(X, c) = \beta(X) \cdot c$$

such that

(1) $\beta(X + X') \cdot c = \beta(X) \cdot c + \beta(X') \cdot c$,

(2) $\beta(X \underline{\times} X') \cdot c = (\beta(X) \cdot c)(\beta(Y) \cdot c)$,

(3) $\beta(X \vee Y) \cdot c = \beta(X) \cdot (\beta(Y) \cdot c)$,

(4) $\beta(e) \cdot c = c$,

where X, X', Y are elements of P. Both X and X' are Σ_r-sets for the same $r \in \mathbb{N}$. And e is the Σ_1-set of one element.

A priori Definition 5.3 is weaker than Definition 5.1.

Let Γ be a finite group. We define in $B(\Gamma)$ the β-operations

$$\beta(X) \cdot c = X \cdot c$$

where $X \in P$, $c \in B(\Gamma)$ and $X \cdot c$ is as $a \cdot c$ above.

Rymer proves in [12, Theorem 2] that $B(\Gamma)$ is a β-ring with Definition 5.3, not 5.1. With our definition of the β-operations it is not necessary to use the structure of compact Lie group in order to prove that theorem, because, by [5, Lemma (4.6)], we can suppose c is a Γ-set and then the proof is similar to Rymer's one.

Observe that this answers the question of Boorman [2, p. 149] without using compact Lie groups.

Finally, $B(S)$ is a β-ring (both with Definitions 5.1 and 5.3) with the action

$$\beta(X) \cdot c = X \vee c \quad (X \in P,\ c \in B(S)).$$

To prove that $B(S)$ is precisely the free β-ring in one generator (with Definition 5.3) is equivalent to show the equivalence between Definitions 5.1 and 5.3. We do not know whether this is true or not; our opinion is that $B(S)$ is, in fact, the free β-ring in one generator.

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